# Landau and Lifshitz Mechanics
# Self Study Notes

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# Contents

# Chapter 1

# The Lagrangian

## 1.1 Notes on Chapter 1

Chapter one deals with the equations of motions and how they can be derived using the principle of least action. As stated by Landau and Lifshitz, "every mechanical system is characterized by a definite function $L(q, \dot{q}, t)$ this allows for even the most complex of systems to be described in a concise way.
Using the action, and finding the extrema one arrives at the Euler-Lagrange equation which can be used to derive the equations of motions from the co-ordinates and velocities of a system. This is shown as follows:
Of particular interest is the fact that we can consider two Lagrangian's $L$ and $L'$ that only differ by a total time derivative. It is the case then that the actions associated with the two Lagrangian must be equal as upon variation the term is destroyed. To shown an example of this consider the E-L equation on some Lagrangian $L$ and add a total time derivative to the equation:

$$L' = L + \frac{dF}{dt} \tag{1.1}$$

Applying Euler-Lagrange on our $\frac{dF}{dt}$ term produces the following:

$$\frac{d}{dt}\frac{\partial \dot{q}}{\partial \dot{q}} - \frac{\partial \dot{q}}{\partial q} = \frac{d1}{dt} = 0 \tag{1.2}$$

Therefore it is clear the equations of motion remain unchanged as the Euler-Lagrange equation takes care of the total time derivative term.

## 1.2 Problems

**Problem 1** Given a double pendulum of two masses $m_1$ and $m_2$ find the Lagrangian of the system.

In order to find the Lagrangian we must determine the generalized co-ordinates and velocities of the system, once these are determined all aspects of the system are known from a classical standpoint. The system given has co-ordinates in x and y for both masses, therefore we will arrive at four equations to describe how the system moves in space: $x_1$, $x_2$, $y_1$, $y_2$. The Lagrangian also requires that we know how the velocities of these quantities behave in time, therefore we must also find $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_1$, $\dot{y}_2$
First we look to the first mass $m_1$ of the double pendulum. The mass $m_1$ changes position in x as a function of $\theta_1(t)$ therefore we can map the x value of $m_1$, defined as $x_1$, using the trigonometric relationship $sin(\theta_1) = \frac{x_1}{l_1}$
Therefore,

$$x_1 = l_1 sin(\theta_1) \tag{1.3}$$

It is clear then that our second mass will follow the same form relative to $\theta_2$ and by the law of superposition will be additive to the displacement due to $x_1$. Therefore,

$$x_2 = l_2 sin(\theta_2) + l_1 sin(\theta_1) \tag{1.4}$$

Now one must consider how the system changes in terms of y for both masses. Consider $m_1$ using the trigonometric relation, $cos(\theta_1) = \frac{y_1}{l_1}$ and knowing that y is defined to be in the positive upwards direction we find:

$$y_1 = -l_1 cos(\theta_1) \tag{1.5}$$

The same process can be applied for $m_2$ and by the law of superposition will be additive to the displacement due to $y_1$ leading to:

$$y_2 = -l_2 cos(\theta_2) - l_1 cos(\theta_1) \tag{1.6}$$

We now have a generalized set of co-ordinates in space given by equations 1.1-1.4. Now one must find the velocities for each parameter in space. Starting with $x_1$ and taking its derivative with respect to time, one yields:

$$\dot{x}_1 = l_1 \dot{\theta}_1 cos(\theta_1) \tag{1.7}$$

Continuing with $x_2$ we find that by taking the time derivative and applying the law of superposition to add the velocity due to $x_1$ we find the following to hold true:

$$\dot{x}_2 = l_2 \dot{\theta}_2 cos(\theta_2) + l_1 \dot{\theta}_1 cos(\theta_1) \tag{1.8}$$

Now we find the velocities for $y_1$ and $y_2$ the same way. leading to the following two equations:

$$\dot{y}_1 = l_1 \dot{\theta}_1 sin(\theta_1) \tag{1.9}$$

$$\dot{y}_2 = l_2 \dot{\theta}_2 sin(\theta_2) + l_1 \dot{\theta}_1 sin(\theta_1) \tag{1.10}$$

Now that we have the velocities described by equations 1.5-1.8 and spatial co-ordinates described by 1.1-1.4 of our system we can construct our kinetic and potential energies of the system to form the Lagrangian of the system. Let us start with the potential U by adding the gravitational potential energy due to $m_1$ and the gravitational potential energy of $m_2$.

$$U = m_1 g y_1 + m_2 g y_2 \tag{1.11}$$

After applying the values for $y_1$ and $y_2$ yields:

$$U = m_1 g[-l_1 cos(\theta_1)] + m_2 g[-l_2 cos(\theta_2) - l_1 cos(\theta_1)] \tag{1.12}$$

Which simplifies to:

$$U = -(m_1 + m_2) g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2) \tag{1.13}$$

Next the Lagrangian requires us to find the kinetic energies of our system which can be described as follows:

$$T = KE_{m_1} + KE_{m_2} \tag{1.14}$$

Which can be written for this system as:

$$T = \frac{1}{2} m_1 (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 (\dot{x_2}^2 + \dot{y_2}^2) \tag{1.15}$$

The kinetic energies associated with $m_1$ in x and y are easily expanded yielding:

$$KE_{m_1} = \frac{1}{2} m_1 [\dot{\theta_1}^2 l_1^2 cos^2(\theta_1) + \dot{\theta_1}^2 l_1^2 sin^2(\theta_1)] \tag{1.16}$$

The kinetic energies associated with $m_2$ are more algebraically complex, but the process remains the same. Let us start by finding $\dot{x_2}^2$ and expand out the terms:

$$\dot{x_2}^2 = \dot{\theta}_2 l_2^2 cos^2(\theta_2) + 2\dot{\theta}_2 l_2 cos(\theta_2) \dot{\theta}_1 l_1 cos(\theta_1) + \dot{\theta}_1 l_2^2 cos^2(\theta_1) \tag{1.17}$$

Now we do the same for finding the $\dot{y_2}^2$ term:

$$\dot{y_2}^2 = \dot{\theta}_2 l_2^2 sin^2(\theta_2) + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_2) sin(\theta_1) + \theta_1 l_1^2 sin^2(\theta_1) \tag{1.18}$$

Now I will make the following substitutions to make the math more clean and easier to simplify:

$$\Phi = \dot{\theta}_2 l_2 \tag{1.19}$$
$$\Omega = \dot{\theta}_1 l_1 \tag{1.20}$$
$$\tag{1.21}$$

We will also use cosine difference identity:

$$cos(\theta_2 - \theta_1) = cos(\theta_2)cos(\theta_1) + sin(\theta_2)sin(\theta_1) \tag{1.22}$$

Expanding the equation and simplifying according to the substitutions defined above we arrive at the following kinetic energy form $m_2$.

$$KE_{m_2} = \frac{1}{2}m_2[\Phi^2[cos^2(\theta_2)+sin^2(\theta_2)]+\Omega^2[cos^2(\theta_1)+sin^2(\theta_1)]+2\Phi\Omega cos(\theta_2-\theta_1)] \tag{1.23}$$

We finally arrive at our Lagrangian:

$$L = T - U = [KE_{m_1} + KE_{m_2}] - U \tag{1.24}$$